Assessment 1 for 41076: Methods in Quantum Computing

due at 11:59 pm of 06 Sep 2021
30 regular + 4 bonus points corresponding to 30% of the overall grade.

1 Quantum circuit identities (10 points)

Let us consider U to be a 2-by-2 unitary matrices such that $\det U = 1$.

1. Let us write $U = (\vec{u}\ \vec{v})$ where $\vec{u}, \vec{v}$ correspond to the columns of matrix U. Show that if U is unitary, $\vec{u}, \vec{v}$ must be orthonormal (both vectors have norm 1 and are orthogonal to each other). Similarly, show that its rows are also orthonormal. This means that if we write $U = \begin{pmatrix} \vec{a}^T \\ \vec{b}^T \end{pmatrix}$ where $\vec{a}^T, \vec{b}^T$ are row vectors, $\vec{a}^T, \vec{b}^T$ are also orthonormal.

2. Show that any 2-by-2 unitary matrix with determinant 1 can be expressed as

$$\begin{pmatrix} e^{i(\alpha/2+\beta/2)}\cos\frac{\theta}{2} & e^{i(\alpha/2-\beta/2)}\sin\frac{\theta}{2} \\ -e^{i(-\alpha/2+\beta/2)}\sin\frac{\theta}{2} & e^{i(-\alpha/2-\beta/2)}\cos\frac{\theta}{2} \end{pmatrix} \tag{1}$$

where α, β, θ are real-valued.

3. Define single qubit rotation gates

$$R_z(\phi) = \begin{pmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{pmatrix} \quad R_y(\phi) = \begin{pmatrix} \cos\frac{\phi}{2} & \sin\frac{\phi}{2} \\ -\sin\frac{\phi}{2} & \cos\frac{\phi}{2} \end{pmatrix}. \tag{2}$$

Show that the matrix in Eq. (1) can be implemented using three rotations as $R_z(\alpha)R_y(\theta)R_z(\beta)$.

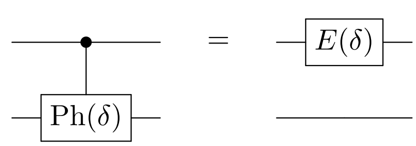

Figure 1: Prove that controlled global phase can be written as a single qubit gate on the control register.

4. Define $\mathrm{Ph}(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & e^{i\delta} \end{pmatrix}$ to be a global phase gate (also known as phase shift). Show that there is always a gate $E(\delta)$ such that the identity in Fig. 1 holds.

2 Working with pure and mixed states (7 points)

Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B$.

1. Compute the density matrix $\sigma = |\psi\rangle\langle\psi|$.
2. Compute the purity $\mathrm{Tr}(\sigma^2)$. Is σ pure?
3. Compute the density matrix of Alice's state $\sigma_A = \mathrm{Tr}_B(\sigma)$.
4. Compute the purity of Alice's state. Is Alice's state pure?
5. Is $|\psi\rangle$ an entangled state and why/why not?

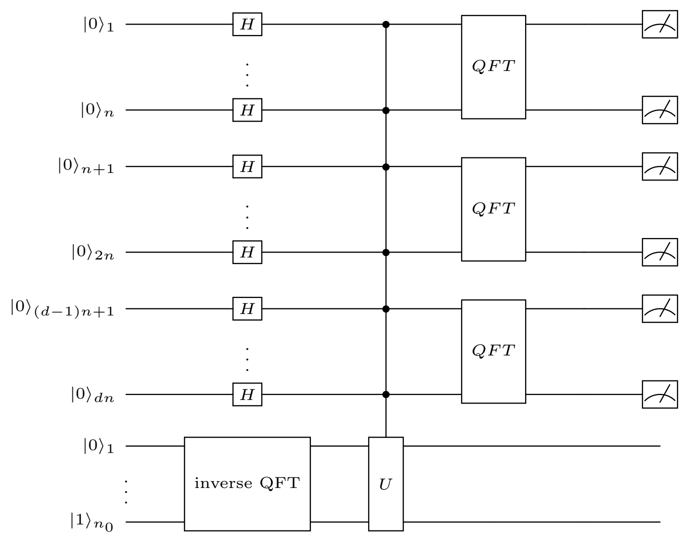

Figure 2: A quantum gradient algorithm by Jordan.

3 Complexity of a quantum algorithm (7 points)

The goal of this problem is to estimate the gate complexity and query complexity of a quantum algorithm for computing gradient. Let $f : \mathcal{R}^d \to \mathcal{R}$ be a function of d variables. We would like to estimate its gradient $\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \ldots, \frac{\partial f}{\partial x_d}\right)$ up to an error $\frac{1}{\epsilon^2}$.

1. First, we estimate the complexity of a classical algorithm. For each component of the gradient $\frac{\partial f}{\partial x_i}$, we evaluate the function f in points $f(x_0, \ldots, x_i + \epsilon/2, \ldots, x_d)$ and $f(x_0, \ldots, x_i - \epsilon/2, \ldots, x_d)$ and compute $\frac{\partial f}{\partial x_i} \approx \frac{f(x_0,\ldots,x_i+\epsilon/2,\ldots,x_d) - f(x_0,\ldots,x_i-\epsilon/2,\ldots,x_d)}{\epsilon}$. How many queries to f does this algorithm use? A tight bound in big-O notation is sufficient.
2. A quantum algorithm accesses f through an oracle O_f such that $O_f|x_1, \ldots, x_d\rangle|z\rangle = |x_1, \ldots, x_d\rangle|z \oplus f(x)\rangle$ where each register x_i consist of n qubits and z a real number encoded into n_0 qubits. The circuit for approximating the gradient is depicted in Fig. 2. It uses d input register with n qubits each and an output register with n_0 qubits. First, it performs a Hadamard transform

on the input registers and inverse quantum Fourier transform (QFT will be covered in Lecture 4 but you can also find an explanation on Wikipedia) on the output register. Next, we apply the oracle and lastly apply QFT on each input register. Compute the gate and query complexity of the algorithm. Big-O asymptotic result in terms of d, n and n_0 is sufficient.

While this algorithm is indeed very interesting, it does not have a lot of applications in quantum computing because the oracle O_f is very powerful and its implementation often wipes out the speedup.

4 Open ended problem (10 points)

Pick a wrong or a misleading claim about quantum computing/mechanics/information/computing, theory or experiment and explain the issues with this claim. The claim can be from a respected media publication (NY Times yes, Medium no), a publicly available claim from a quantum scientist or CEO/VP of a quantum company. Use references to quantum literature to support your argument and acknowledgements when appropriate. Note that your argument does not need to be original, only well-researched and articulated. The minimum length is 100 words and maximum 2 pages, excluding references. Please try to find a different claim than your classmates.