Solution to exercise 9 in lecture notes 3-4.

We can write ρ, σ in the basis where they are diagonal:

$$\rho = \sum_i \rho_i |i\rangle\langle i| \qquad \sqrt{\rho} = \sum_i \sqrt{\rho_i}\, |i\rangle\langle i| \qquad \rho_i, \sigma_\alpha \geq 0$$

$$\sigma = \sum_\alpha \sigma_\alpha |\alpha\rangle\langle\alpha| \qquad \sqrt{\sigma} = \sum_\alpha \sqrt{\sigma_\alpha}\, |\alpha\rangle\langle\alpha|$$

$|i\rangle, |\alpha\rangle$ are not orthogonal!

We can also use an alterative form for fidelity and show that it is equivalent to our definition:

we used $|A| = \sqrt{A^\dagger A}$

$$F(\sigma,\rho) = \left(\mathrm{Tr}\, |\sqrt{\sigma}\sqrt{\rho}|\right)^2$$

$$= \left(\mathrm{Tr} \sqrt{(\sqrt{\sigma}\sqrt{\rho})^\dagger (\sqrt{\sigma}\sqrt{\rho})}\right)^2$$

$$= \left(\mathrm{Tr} \sqrt{(\sqrt{\rho})^\dagger (\sqrt{\sigma})^\dagger \sqrt{\sigma}\sqrt{\rho}}\right)^2$$

$$= \left(\mathrm{Tr} \sqrt{\sqrt{\rho}\ \sigma\ \sqrt{\rho}}\right)^2$$

we can see the next step from the decomposition of $\sqrt{\rho}, \sqrt{\sigma}$

As a norm, $\mathrm{Tr}\, |\sqrt{\sigma}\sqrt{\rho}| \geq 0$. 9.1

we can show symmetry by taking

$$\mathrm{Tr}\, |\sqrt{\sigma}\sqrt{\rho}| = \mathrm{Tr}\sqrt{\sqrt{\sigma}\sqrt{\rho}\,(\sqrt{\sigma}\sqrt{\rho})^\dagger} = \mathrm{Tr}\sqrt{\sqrt{\sigma}\,\rho\,\sqrt{\sigma}} \qquad 9.4$$

We can then use Holder's inequality:

$$\|AB\|_1 \leq \|A\|_p \, \|B\|_q \quad \text{where } \tfrac{1}{p} + \tfrac{1}{q} = 1 \text{ and } p,q \geq 1 \text{ for } p=q=2$$

$$\|\sqrt{\sigma}\sqrt{\rho}\|_1 \leq \|\sqrt{\sigma}\|_2 \, \|\sqrt{\rho}\|_2$$

$$= \sqrt{\sum_\alpha \sigma_\alpha}\ \sqrt{\sum_i \rho_i} = 1 \qquad 9.1$$

We used the result of Exercise 4

F is $\|\sqrt{\sigma}\sqrt{\rho}\|_1^2$ which is also ≤ 1.

9.2. We use the fact that $\sqrt{U\rho U^\dagger} = U\sqrt{\rho}\,U^\dagger$ (and the same for σ). To see that:

$$(U\rho U^\dagger)^2 = U\rho U^\dagger U\rho U^\dagger$$
$$= U\rho^2 U$$

Then:

$$\mathrm{Tr}\sqrt{\sqrt{U\rho U^\dagger}\,U\sigma U^\dagger\sqrt{U\rho U^\dagger}} = \mathrm{Tr}\sqrt{U\sqrt{\rho}\,U^\dagger U\sigma U^\dagger U\sqrt{\rho}\,U^\dagger}$$
$$= \mathrm{Tr}\sqrt{U\sqrt{\rho}\,\sigma\sqrt{\rho}\,U^\dagger}$$
$$= \mathrm{Tr}\,U\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\,U^\dagger$$
$$= \mathrm{Tr}\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}$$

9.3 $$F(|\psi_\rho\rangle, \sigma) = \left(\mathrm{Tr}\sqrt{\sqrt{|\psi_\rho\rangle\langle\psi_\rho|}\,\sigma\sqrt{|\psi_\rho\rangle\langle\psi_\rho|}}\right)^2$$
$$= \left(\mathrm{Tr}\sqrt{|\psi_\rho\rangle\langle\psi_\rho|\sigma|\psi_\rho\rangle\langle\psi_\rho|}\right)^2$$
$$= \langle\psi_\rho|\sigma|\psi_\rho\rangle\left(\mathrm{Tr}\sqrt{|\psi_\rho\rangle\langle\psi_\rho|}\right)^2$$
$$= \langle\psi_\rho|\sigma|\psi_\rho\rangle$$

If one of the states is pure this is a useful expression

$$F(|\psi_\rho\rangle, |\psi_\sigma\rangle) = \langle\psi_\rho|\psi_\sigma\rangle\langle\psi_\sigma|\psi_\rho\rangle = |\langle\psi_\rho|\psi_\sigma\rangle|^2$$

An alternative way for showing these properties is through Ulman's theorem (see Nielsen Chuang)