POVM example

$$\sum_i M_i = \mathbb{1} \checkmark$$

all eigenvalues were 0, 1/3

→ just calculate them

$$M_5 = \frac{1 \pm Z}{6} = \frac{|0\rangle\langle 0| + \cancel{|1\rangle\langle 1|} \pm |0\rangle\langle 0| - \cancel{|1\rangle\langle 1|}}{6}$$

$$= \frac{1}{3}|0\rangle\langle 0| \rightarrow \lambda = \frac{1}{3}, 0 \qquad H\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}H$$

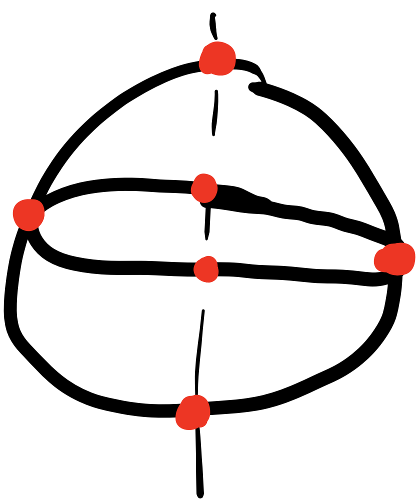
$\rightarrow \frac{1}{3}|1\rangle\langle 1|$

→ $\mathbb{1}$ is $\mathbb{1}$ in any basis

$$M_1 = \frac{\mathbb{1} + X}{2} = \frac{(|+\rangle\langle +| + |-\rangle\langle -|) + (\overset{+1}{|+\rangle\langle +|} - \overset{-1}{|-\rangle\langle -|})}{6}$$

$$= \frac{1}{3}\left(|+\rangle\langle +|\right)$$

⋮



$$\frac{\text{projectors on these points}}{3}$$

$$P_{1_2} = \mathrm{Tr}\left(\rho M_1^{M_2}\right)$$

$$= \mathrm{Tr}\left[\frac{1}{2}\left(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\right)\frac{1}{6}\left(\mathbb{1} \pm \overset{\sigma_x}{X}\right)\right]$$

$$= \mathrm{Tr}\Big[\frac{1}{12}\big(\mathbb{1} + \cancel{r_0\sigma_x} + \cancel{r_1\sigma_y} + \cancel{r_2\sigma_z} +$$

$$\pm \cancel{X} \pm r_0 X^2 \pm r_2\cancel{\sigma_y\sigma_x} \pm r_2\cancel{\sigma_z\sigma_x}\big)\Big]$$

this is $\sim Y$

$\sim Z$

$$\boxed{\mathrm{Tr}(X) = \mathrm{Tr}(Y) = \mathrm{Tr}(Z) = 0}$$

$$= \frac{1}{12}\cdot 2 \pm \frac{r_0}{12}\cdot 2$$

$$P_1 = \frac{1}{6} + \frac{r_0}{6}$$

$$P_2 = \frac{1}{6} - \frac{r_0}{6}$$

For P_2 we only changed a few signs (see above in blue).

We simplified the calculation by realizing that X, Y, Z are traceless.

→ Therefore, for r_0 we get

$$P_1 - P_2 = \frac{r_0}{3} \Rightarrow r_0 = 3(P_1 - P_2)$$

$$P_{3_{4}} = \mathrm{Tr}\,(\rho M_{3_{4}})$$ same calculation

$$= \mathrm{Tr}\left[\left(\tfrac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma}\right)\tfrac{1}{6}\,(\mathbb{1} \pm Y)\right]$$

$$= \frac{1}{12}\,\mathrm{Tr}\Big[\mathbb{1} \pm Y + r_0 X \pm r_0 XY + r_1 Y \pm r_1 \underbrace{YY}_{\mathbb{1}} +$$

$$+ r_2 Z \pm r_2 ZY\Big]$$

$$= \frac{1}{12}\,(2 \pm 2r_1) = \frac{1}{6}\,(1 \pm r_1)$$

$$P_3 - P_4 = \frac{r_1}{3} \qquad r_1 = 3(P_3 - P_4)$$

$$P_{5_{6}} = \mathrm{Tr}\,(\rho M_{5_{6}})$$

$$= \mathrm{Tr}\left[\tfrac{1}{2}(\mathbb{1} + \vec{r}\cdot\vec{\sigma})\tfrac{1}{6}\,(\mathbb{1} \pm Z)\right]$$

$$= \frac{1}{6}\,(1 \pm r_2) \Rightarrow r_2 = 3\,(P_5 - P_6)$$