Exercise

Verify that Pauli X is a Hermitian operator and compute its eigenvalues and eigenvectors.

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\det(X - \lambda \mathbb{1}) = 0 \quad \lambda \text{ is eigenval.}$$

$$\begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$$

$$\underline{\underline{\lambda = \pm 1}}$$

eigenvalues

X is real

$$X^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$$

It is Hermitian.

$$X \vec{u} = \lambda \vec{u} \quad \text{eigenvector}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \overset{+1}{=} \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} b = a \\ a = b \\ |a|^2 + |b|^2 = 1 \end{matrix}$$

$$\rightarrow u_{+1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \overset{-1}{=} - \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow \begin{matrix} b = -a \\ a = -b \\ |a|^2 + |b|^2 = 1 \end{matrix}$$

$$\rightarrow u_{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Exercise

- Cyclic property: Show that $\operatorname{Tr} LM = \operatorname{Tr} ML$.
- Show that $\operatorname{Tr} A$ is independent of the basis of A.

Basis $|x\rangle$

$\mathrm{Tr}(A) = \sum_x \langle x|A|x\rangle$

$|x'\rangle = U|x\rangle$ new basis

$\mathrm{Tr}(A) = \sum_{x'} \langle x'|A|x'\rangle$

$= \sum_x \langle x|U^\dagger A U|x\rangle$

$= \sum_x \langle x|A U U^\dagger|x\rangle$

$= \sum_x \langle x|A|x\rangle$

$L = \sum_{ij} L_{ij} |i\rangle\langle j| \qquad M = \sum_{kl} M_{kl} |k\rangle\langle l|$

$$\mathrm{Tr}(LM) = \sum_m \sum_{\substack{ij\\kl}} \underbrace{L_{ij} M_{kl}}_{\text{scalars}} \underbrace{\langle m|i\rangle}_{\delta_{mi}} \underbrace{\langle j|k\rangle}_{\delta_{jk}} \underbrace{\langle l|m\rangle}_{\delta_{lm}}$$

$$= \sum_{m,j} L_{mj} M_{jm}$$

same

same

$$\mathrm{Tr}(ML) = \sum_m \sum_{\substack{ij\\kl}} M_{kl} L_{ij} \underbrace{\langle m|k\rangle}_{\delta_{m,k}} \underbrace{\langle l|i\rangle}_{\delta_{l,i}} \underbrace{\langle j|m\rangle}_{\delta_{j,m}}$$

scalars commute

$$= \sum_{m,l} M_{ml} L_{lm} = \begin{bmatrix}\text{Substitute}\\ l \to j\end{bmatrix} = \sum_{m,j} L_{mj} M_{jm}$$

Exercise

Let $|\Phi\rangle_{AB} = \frac{1}{\sqrt{2}}(|0\rangle_A \otimes |0\rangle_B + |1\rangle_A \otimes |1\rangle_B)$. Compute $\mathrm{Tr}_A(|\Phi\rangle\langle\Phi|_{AB})$ and $\mathrm{Tr}_B(|\Phi\rangle\langle\Phi|_{AB})$.

$\langle\Phi|_{AB} = \frac{1}{\sqrt{2}}(\langle 0|_A \otimes \langle 0|_B + \langle 1|_A \otimes \langle 1|_B)$

$|\Phi\rangle\langle\Phi|_{AB} = \frac{1}{2}\big(|0\rangle\langle 0|_A \otimes |0\rangle\langle 0|_B + |0\rangle\langle 1|_A \otimes |0\rangle\langle 1|_B + |1\rangle\langle 0|_A \otimes |1\rangle\langle 0|_B + |1\rangle\langle 1|_A \otimes |1\rangle\langle 1|_B\big)$

Compute $\mathrm{Tr}_A(\cdot)$ for each term

$\mathrm{Tr}_A(|0\rangle\langle 0| \otimes |0\rangle\langle 0|) = \langle 0|0\rangle\, |0\rangle\langle 0|_B = |0\rangle\langle 0|$

$\mathrm{Tr}_A(|1\rangle\langle 1| \otimes |1\rangle\langle 1|) = \langle 1|1\rangle\, |1\rangle\langle 1|_B = |1\rangle\langle 1|$

$\mathrm{Tr}_A(|0\rangle\langle 1| \otimes |0\rangle\langle 1|) = \langle 0|1\rangle\, |0\rangle\langle 1|_B = 0$

$\mathrm{Tr}_A(|1\rangle\langle 0| \otimes |1\rangle\langle 0|) = \langle 1|0\rangle\, |1\rangle\langle 0|_B = 0$

Together $\mathrm{Tr}_A(|\Phi\rangle\langle\Phi|_{AB}) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

From symmetry, Tr_B will be the same.

Exercise

There are three necessary and sufficient criteria that a matrix corresponds to a valid description to a quantum state. Show that

probabilities $0 \leq p_i \leq 1$

$$\rho := \sum_i p_i |\psi_i\rangle\langle\psi_i|, \qquad \sum_i p_i = 1 \tag{1}$$

satisfies all three of them

1. ρ is Hermitian [1] $\rho^\dagger = \sum_i p_i^* |\psi_i\rangle\langle\psi_i| = \rho$
2. ρ is positive semi-definite [2] p_i are eigenvalues and they are all non-negative
3. $\mathrm{Tr}[\rho] = 1$.

In the basis of $\{|\psi_i\rangle\}$, $\mathrm{Tr}[\rho] = \sum_i p_i = 1$

[1] A hermitian matrix A satisfies $A^\dagger = A$.

[2] Eigenvalues of a positive semi-definitive matrix are real and equal to 0 or positive.

Exercise

Define purity of a quantum state as $\mathrm{Tr}[\rho^2]$. Show that unitary operations preserve purity, i.e. a pure state never gets mapped onto a mixed state and vice versa.

Apply a unitary on ρ: $\rho \to U^\dagger \rho U$

$$\mathrm{Tr}\left[(U^\dagger \rho U)(U^\dagger \rho U)\right] = \mathrm{Tr}\left[U^\dagger \rho^2 U\right] = \mathrm{Tr}\left[\rho^2 U U^\dagger\right]$$

$$= \mathrm{Tr}\left[\rho^2\right]$$

No cloning theorem

Theorem (No-Cloning theorem)
There is no unitary operation $U_{\rm copy}$ on $\mathcal{H}_A \otimes \mathcal{H}_B$ such that for all $|\psi\rangle_A \in \mathcal{H}_A$ and $|0\rangle_B \in \mathcal{H}_B$

$$U_{\rm copy}(|\phi\rangle_A \otimes |0\rangle_B) = e^{if(\phi)}|\phi\rangle_A \otimes |\phi\rangle_B \tag{2}$$

for some number $f(\phi)$ that depends on the initial state $|\phi\rangle$.

Proof: in the lecture notes