Problem set 1 for 41076: Methods in Quantum Computing

due August 29th, at 3 pm
15 regular + 3 bonus points

1 Linear algebra refresher (4 points)

1. Define the rotation matrix $R_Z(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$. Show that $XR_Z(\alpha)X = R_Z(-\alpha)$.

2. Consider a Hermitian matrix H. Show that a matrix $U = e^{iH}$ will be unitary.

1.1 $$XR_Z(\alpha)X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & e^{i\alpha/2} \\ e^{-i\alpha/2} & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} e^{-i\alpha/2} & 0 \\ 0 & e^{i\alpha/2} \end{pmatrix} = R_Z(-\alpha)$$

1.2. We will show that U satisfies $U^+U = \mathbb{1}$

$$U^+ = (e^{iH})^+ = \sum_k \left[\frac{(iH)^k}{k!}\right]^+ = \sum_k \frac{((-i)^+ H^+)^k}{k!}$$

We have $(i)^+ = -i$ and $H^+ = H$

$$U^+ = \sum_k \frac{(-iH)^k}{k!} = e^{iH}$$

and $U^+U = e^{-iH} e^{iH} = \mathbb{1} \Rightarrow U$ is unitary

2 Quantum circuits (5 points)

1. Prove the circuit identity in Fig. 1.

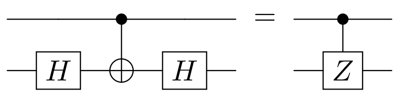


Figure 1: These quantum circuits are equivalent.

2. Suppose we implemented the circuit in Fig. 2 and measured the first register. If we measure the first qubit to be in a state $|0\rangle$, what is the state of the second register?

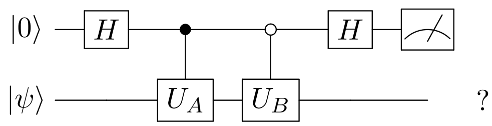


Figure 2: What is the state of the second register if the state of the first register was measured to be $|0\rangle$?

2.1

$$(\mathbb{1}\otimes H)(CNOT)(\mathbb{1}\otimes H) = |0\rangle\langle 0| \otimes H\mathbb{1}H + |1\rangle\langle 1| \otimes HXH =$$

$$= |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes H(|0\rangle\langle 1| + |1\rangle\langle 0|)H = |0\rangle\langle 0| \otimes \mathbb{1} +$$

$$+ |1\rangle\langle 1| \otimes \frac{1}{2}(|0\rangle\langle 0| + \cancel{|1\rangle\langle 0|} - \cancel{|0\rangle\langle 1|} \underbrace{- |1\rangle\langle 1| + |0\rangle\langle 0|}_{2} + \cancel{|0\rangle\langle 1|} - \cancel{|1\rangle\langle 0|} - |1\rangle\langle 1|)$$

$$= |0\rangle\langle 0| \otimes \mathbb{1} + |1\rangle\langle 1| \otimes \frac{1}{\cancel{2}}(\cancel{2}|0\rangle\langle 0| - \cancel{2}|1\rangle\langle 1|) = CZ$$

2.2

$$|0\rangle|\psi\rangle \xrightarrow{H} \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle|\psi\rangle) \xrightarrow{C\text{-}U_A} \frac{1}{\sqrt{2}}(|0\rangle|\psi\rangle + |1\rangle U_A|\psi\rangle)$$

$$\xrightarrow{C\text{-}U_B} \frac{1}{\sqrt{2}}(|0\rangle U_B|\psi\rangle + |1\rangle U_A|\psi\rangle) \xrightarrow{H} \frac{1}{2}(|0\rangle(U_A + U_B)|\psi\rangle +$$

$$+ |1\rangle(U_B - U_A)|\psi\rangle)$$

If we measure $|0\rangle$, we obtained a state proportional to $U_A + U_B|\psi\rangle$, i.e.

$$\frac{U_A + U_B|\psi\rangle}{\| U_A + U_B|\psi\rangle \|}$$

3 Working with pure and mixed states (6 points)

Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$.

1. Verify that $|\langle\psi||\psi\rangle| = 1$
2. Compute the density operator $\sigma = |\psi\rangle\langle\psi|$. You can keep it in the bra-ket formalism or write it as a matrix.
3. Compute the purity $\mathrm{Tr}(\sigma^2)$. Is σ pure?
4. Compute the density operator of Alice's state $\sigma_A = \mathrm{Tr}_B(\sigma)$.
5. Compute the purity of Alice's state. Is Alice's state pure?
6. Is $|\psi\rangle$ an entangled state and why/why not?

3.1. $|\langle\psi|\psi\rangle| = \frac{1}{2}(\langle 00| + \langle 11|)(|00\rangle + |11\rangle) = \frac{1}{2}(\langle 00|00\rangle + \langle 11|11\rangle + 0 + 0$

$= 1$ It is normalized

3.2. $\sigma = |\psi\rangle\langle\psi| = \frac{1}{2}(|00\rangle + |11\rangle)(\langle 00| + \langle 11|) = \frac{1}{2}(|00\rangle\langle 00|$

$|00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|))$.

3.3

$\sigma^2 = \sigma \cdot \sigma = \frac{1}{4}(|00\rangle\langle 00| + |00\rangle\langle 11| + |00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| +$

$+ |11\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|) = \sigma$ $\quad \mathrm{Tr}(\sigma) = \mathrm{Tr}(\sigma^2) = 1$ Yes, pure.

3.4. $\sigma_A = \mathrm{Tr}_B(\sigma) = \frac{1}{2}(|0\rangle\langle 0| \cdot \langle 0|0\rangle + |0\rangle\langle 1| \cdot \langle 0|1\rangle + |1\rangle\langle 0| \langle 1|0\rangle +$

$+ |1\rangle\langle 1| \cdot \langle 1|1\rangle) = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{1}{2}\mathbb{1}$

3.5 $\sigma_A^2 = \frac{1}{4}\mathbb{1}$ $\quad \mathrm{Tr}(\frac{1}{4}\mathbb{1}) = \frac{1}{4} \cdot 2 = \frac{1}{2}$ No, it's not.

Yes, we traced out of a pure state and got a mixed state so the original state was entangled. It is one of the maximally entangled states.

4 Bonus (3 points)

Devise a quantum circuit that implements $|a\rangle|b\rangle|000\rangle \to |a\rangle|b\rangle|a+b\rangle$ where a, b are 2-bit binary numbers, i.e. $a, b \in \{00, 01, 10, 11\}$.

First we implement an adder on single bits:

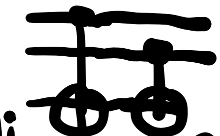
$|0\rangle|0\rangle|00\rangle \to |0\rangle|0\rangle|00\rangle$

$|0\rangle|1\rangle|00\rangle \to |0\rangle|1\rangle|01\rangle$

$|1\rangle|0\rangle|00\rangle \to |1\rangle|0\rangle|01\rangle$

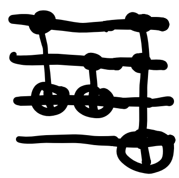
$|1\rangle|1\rangle|00\rangle \to |1\rangle|1\rangle|10\rangle$

each can be implemented with a CNOT



↳ we can use a Toffoli gate.

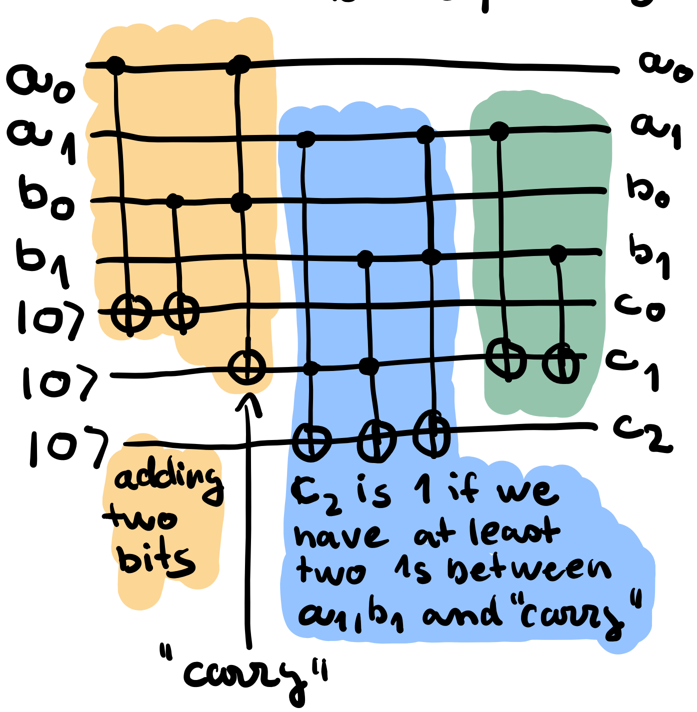
1 bit adder:



This will be the foundation for 2bits but we need to rembember to carry over 1 if needed.

we write $a = a_1 \cdot 2 + a_0$ $\quad c = a + b = c_2 \cdot 4 + c_1 \cdot 2 + c_0$

$b = b_1 2 + a_0$



computing c_1. we are careful to do this last so we didn't forget carry