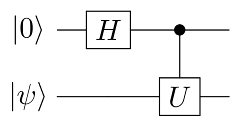
Problem set 2 for 41076: Methods in Quantum Computing

due September 15, 23:59
15 points

1 Quantum channels (5 points)

Consider the following quantum circuit



where U is some single qubit gate and $|\psi\rangle$ is a quantum state on a single qubit.

1. What is the output of this circuit?
2. Now, trace out the first qubit. What is the reduced density matrix corresponding to the second register after the application of the circuit?
3. Looking at the second register only, write the operation corresponding to the circuit as a quantum channel. What are its Kraus operators?
4. Take $U = R_Y(\theta)$ where

$$R_Y(\theta) = \exp(-iY\theta/2) = \begin{pmatrix} \cos(\theta/2) & -\sin(\theta/2) \\ \sin(\theta/2) & \cos(\theta/2) \end{pmatrix} \tag{1}$$

   and $|\psi\rangle = a|0\rangle + b|1\rangle$ where $a, b \in \mathbb{C}$. What is the fidelity between $|\psi\rangle\langle\psi|$ and the state on the second qubit?

2 SIC POVM (5 points)

Consider the following quantum states:

$$|v_1\rangle = |0\rangle \tag{2}$$

$$|v_2\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}|1\rangle \tag{3}$$

$$|v_3\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\frac{2\pi}{3}}|1\rangle \tag{4}$$

$$|v_4\rangle = \frac{1}{\sqrt{3}}|0\rangle + \sqrt{\frac{2}{3}}e^{i\frac{4\pi}{3}}|1\rangle \tag{5}$$

$$\tag{6}$$

1. Define $M = \{M_1, M_2, M_3, M_4\}$ where $M_i = \frac{1}{2}|v_i\rangle\langle v_i|$. Show that M is a POVM.
2. Consider a general quantum state ρ parametrized as

$$\rho = \frac{1}{2}(\mathbb{1} + r_1 X + r_2 Y + r_3 Z) \tag{7}$$

   where X, Y, Z are Paulis. Find how the parameters r_i depend on the expectation values of elements of M. How can we reconstruct ρ from the measurement statistics of M?

3 Spin in a magnetic field (5 points)

Define a hermitian matrix

$$H = -\frac{\mu B}{2m}X \tag{8}$$

where $\frac{\mu B}{2m}$ is real and positive.

1. Find the eigenvalues and corresponding eigenvectors of H? Which one corresponds to the ground state?
2. Consider the state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{9}$$

   What is the expected energy in this state?
3. Starting in state $|\psi(0)\rangle$, find the evolution under H.

4 Bonus (2 points)

Consider the transpose map defined as $T : \rho \to \rho^T$, i.e. mapping a density matrix of a state to its transpose. Show that

1. The map T is trace-preserving and positive, i.e. preserves the trace and semidefinity of ρ.
2. T is not completely positive i.e. there exists a larger state $|\psi\rangle$ ($\rho = Tr_A(|\psi\rangle\langle\psi|)$ and the induced map $\mathbb{I}_A \otimes T_B$ acting on the larger system will not be always positive.