Problem set 1 for 41076: Methods in Quantum Computing

due August 29th, at 3 pm
15 regular + 3 bonus points

1 Linear algebra refresher (4 points)

1. Define the rotation matrix $R_Z(\alpha) = \begin{pmatrix} e^{i\alpha/2} & 0 \\ 0 & e^{-i\alpha/2} \end{pmatrix}$. Show that $XR_Z(\alpha)X = R_Z(-\alpha)$.

2. Consider a Hermitian matrix H. Show that a matrix $U = e^{iH}$ will be unitary.

2 Quantum circuits (5 points)

1. Prove the circuit identity in Fig. 1.

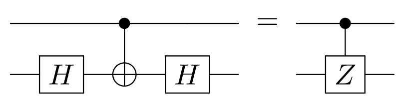

Figure 1: These quantum circuits are equivalent.

2. Suppose we implemented the circuit in Fig. 2 and measured the first register. If we measure the first qubit to be in a state $|0\rangle$, what is the state of the second register?

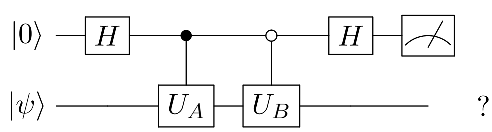

Figure 2: What is the state of the second register if the state of the first register was measured to be $|0\rangle$?

3 Working with pure and mixed states (6 points)

Alice and Bob share the state $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B)$.

1. Verify that $|\langle\psi||\psi\rangle| = 1$

2. Compute the density operator $\sigma = |\psi\rangle\langle\psi|$. You can keep it in the bra-ket formalism or write it as a matrix.

3. Compute the purity $\mathrm{Tr}(\sigma^2)$. Is σ pure?

4. Compute the density operator of Alice's state $\sigma_A = \mathrm{Tr}_B(\sigma)$.
5. Compute the purity of Alice's state. Is Alice's state pure?
6. Is $|\psi\rangle$ an entangled state and why/why not?

4 Bonus (3 points)

Devise a quantum circuit that implements $|a\rangle|b\rangle|000\rangle \rightarrow |a\rangle|b\rangle|a+b\rangle$ where a, b are 2-bit binary numbers, i.e. $a, b \in \{00, 01, 10, 11\}$.