Methods in quantum computing

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Announcements

$|\text{amplitude}|^2$

$a a^* = p$ $\quad a = i$

$p = 1$

- Problem set 1 is due
- I require your full solutions, not just the answers
- If you already submitted a solution but would like to expand on it, correct something, please do so by the end of the day (otherwise a late penalty will apply)
- Solved problem sets will appear on the website, also problem set 2
- Has everyone who submitted set 0 received an email from me?

positive semidefinite matrix

$\Leftrightarrow$ all eigenvalues are larger or equal to 0 , $X \to \lambda = \pm 1 \to$ NOT

Fidelity and Trace distance

Fidelity $F(\rho, \sigma) := \left(\mathrm{Tr}\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)^2$ IN Nielsen, this 2 is not used

If at least one of the states is pure $F = \langle\psi|\sigma|\psi\rangle$.

Can be interpreted as measuring σ with POVM $M_0 = |\psi\rangle\langle\psi|$, $M_1 = \mathbb{I} - |\psi\rangle\langle\psi|$.

Trace distance $Tr(A, B) = \frac{1}{2}\mathrm{Tr}\underbrace{|A - B|}$ - tells us how easy it is to distinguish between A and B, more difficult to measure.

Infidelity and trace distance on pure state

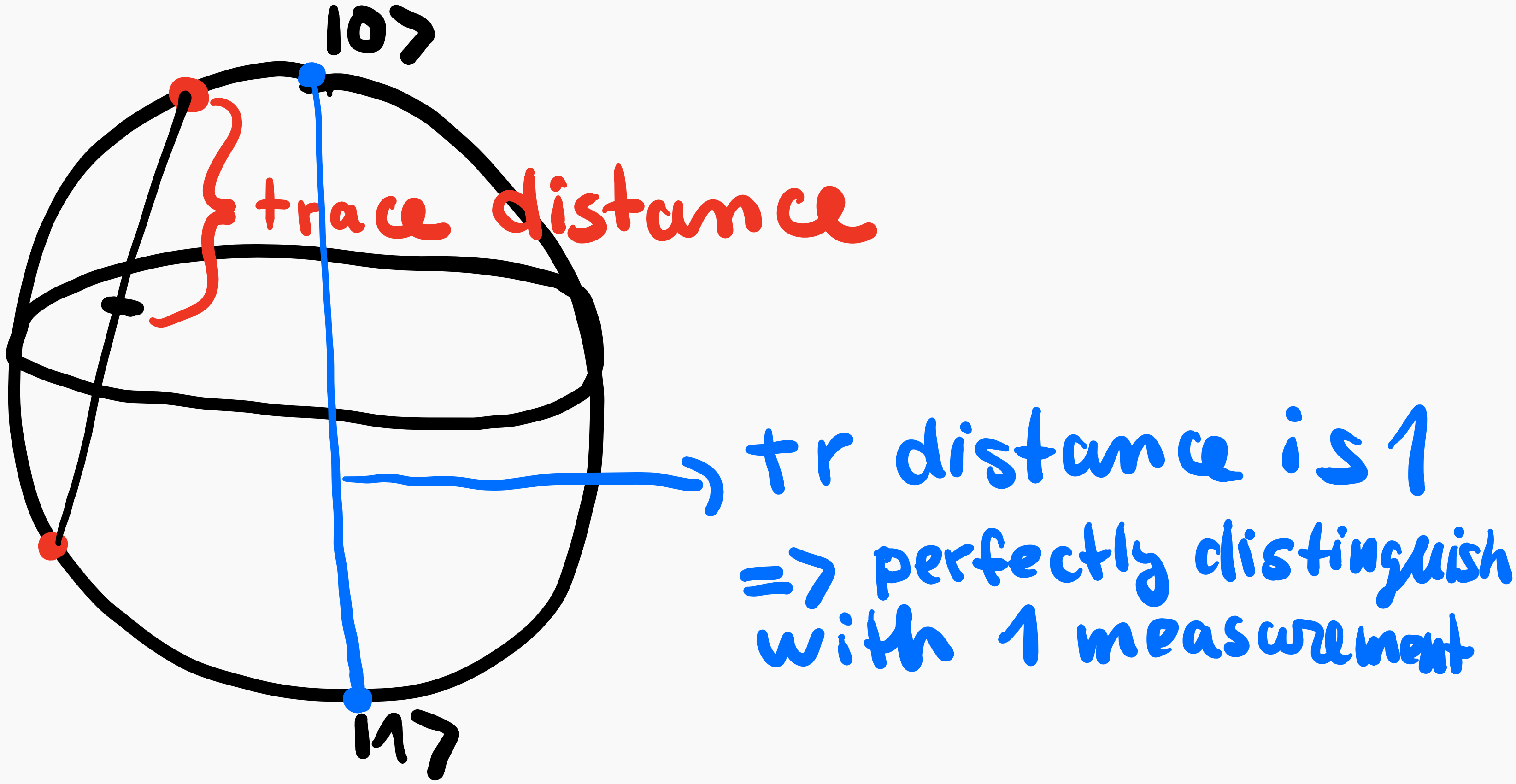
Infidelity: $1 - |\langle\psi|\phi\rangle|^2$ 1 - fidelity

Trace distance: $\sqrt{1 - |\langle\psi|\phi\rangle|^2}$

mixed state tr-distance it's

not $\cancel{\sqrt{1 - \langle\psi|\sigma|\psi\rangle}}$

Geometric representation

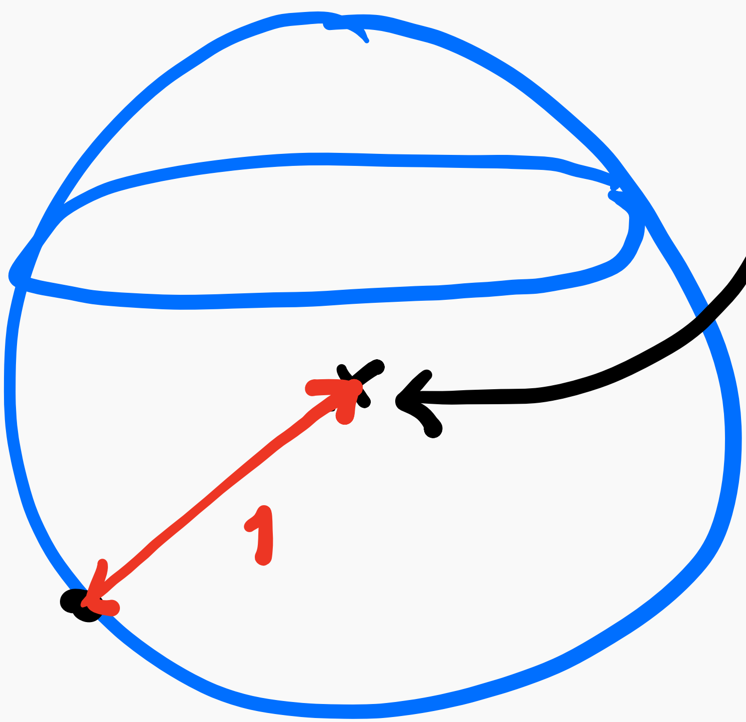


Exercise

1. $0 \leq F(\rho, \sigma) \leq 1$.
2. $F(U\rho U^\dagger, U\sigma U^\dagger) = F(\rho, \sigma)$.
3. $F(|\psi_\rho\rangle, |\psi_\sigma\rangle) = |\langle\psi_\rho|\psi_\sigma\rangle|^2$.
4. Symmetry: $F(\rho, \sigma) = F(\sigma, \rho)$.

Exercise

What is the fidelity and total trace distance between a maximally mixed state and any pure state?



maximally mixed state

$$\rho_I = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\mathrm{Tr}(|\psi\rangle\langle\psi|, \rho_I) = \frac{1}{2}$$

$$\text{Fidelity} = \langle\psi|\rho|\psi\rangle = \frac{1}{2}\langle\psi|\psi\rangle = \frac{1}{2}$$

$$\text{Infidelity} = \frac{1}{2}$$

Quantum channels

previously:

Kraus operators

$$\Phi(\sigma) = \sum_i B_i \sigma B_i^\dagger \quad \text{where} \quad \sum_i B_i^\dagger B_i = \mathbf{1}. \tag{1}$$

Examples of channels

- Depolarizing Channel:

$$\mathcal{N}(\rho) = (1 - p)\rho + p\pi,$$

prob. p of error X, Y or Z

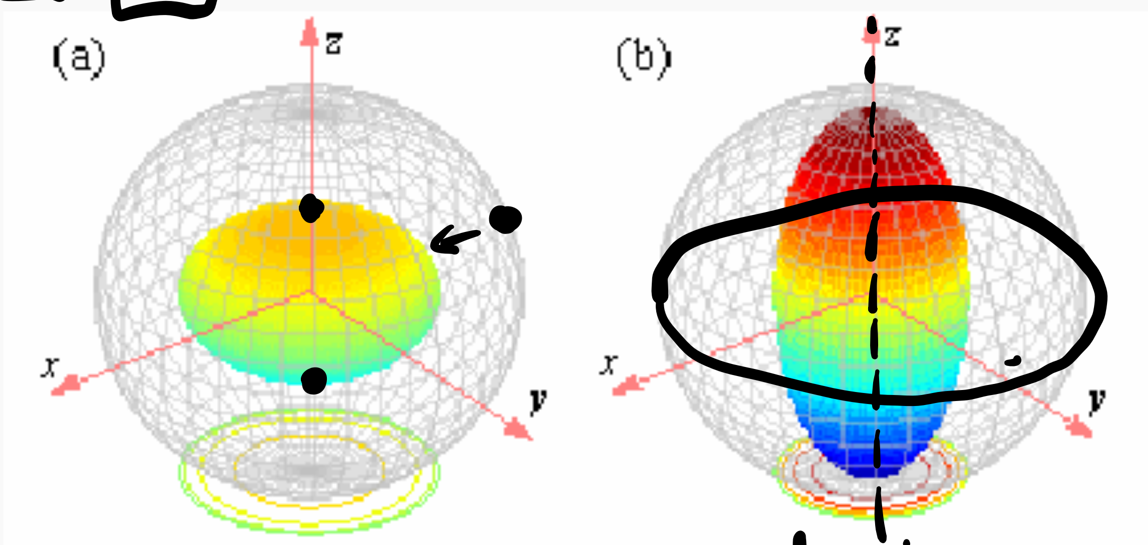
where π is the completely mixed state.

- Dephasing Channel:

$$\mathcal{N}(\rho) = (1 - p)\rho + pZ\rho Z.$$

$|0\rangle - \boxed{U} - \boxed{\mathcal{N}} - \boxed{U} - \boxed{\mathcal{N}} - \cdots$

$Z|+\rangle\langle+|Z \to |-\rangle\langle-|$



depol. dephase

Average fidelity of a quantum channel

Defined with respect to the identity channel

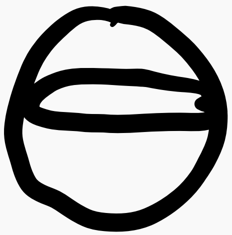
without noise

$$F(\mathcal{E}) = \int d\psi \, \langle\psi| \, \mathcal{E}(\psi) \, |\psi\rangle \tag{2}$$

with noise

as an average over all state fidelities. To obtain the average, we must integrate over all the quantum states in a given Hilbert space with equal weightings and satisfy $\int d\psi = 1$. This is known as integration over Haar measure.

which state?



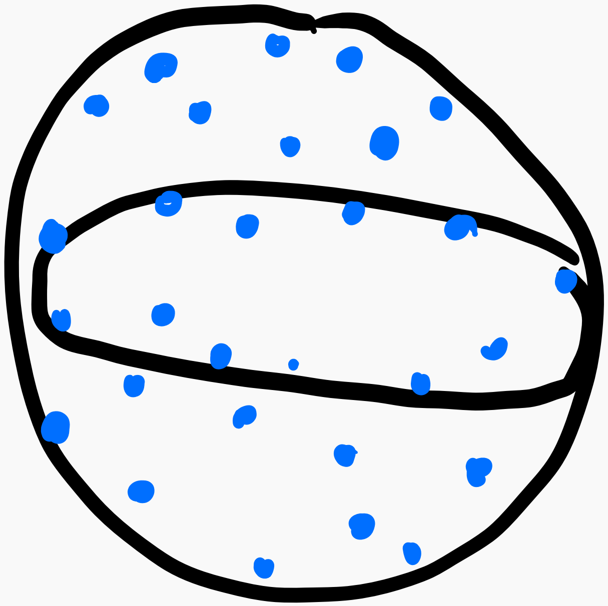
integrate over the sphere

Exercise

Compute the fidelity of a qubit depolarizing channel

$\mathcal{E}(\rho) = (1-p)\,|\psi\rangle\langle\psi| + p\frac{I}{d}$.

d-dimensions



→ in practice
evaluate through
sampling
↳ from what?
set of state that
approximate random
quantum states (t-design)
→ RANDOMIZED BENCHMARKING

Unitary channel

The special case of fidelity between a unitary and the identity can be simplified through the Nielsen's equation:

$$F(U) = \frac{d + |Tr(U)|^2}{d + d^2}. \tag{3}$$

Exercise

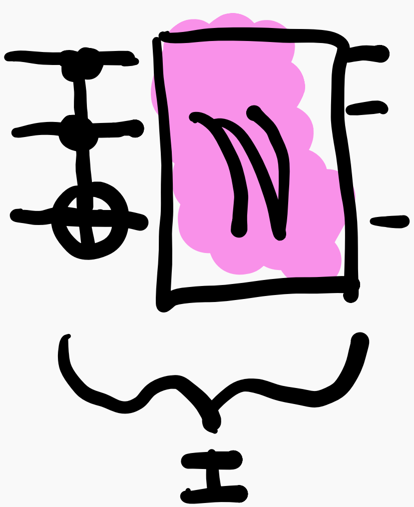
1. Verify that $F(I)=1$ in (3).

2. The hottest quantum startup promises to do quantum computing by implementing Hadamard and Toffoli gates. However, they have a minor issue: their Toffoli gates are not working and they are simply doing nothing (i.e. identity gates). What is the fidelity of their "Toffoli" gate?

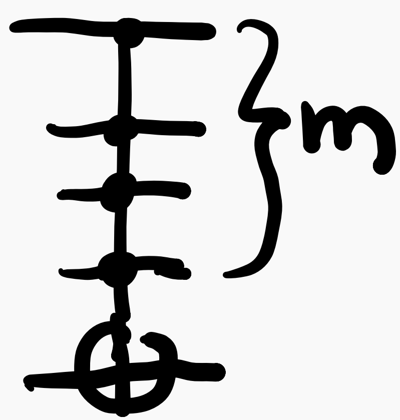
$$\frac{8+6^2}{8+8^2}$$

3. What if they replace all m-controlled-NOT gates with the identity?

$$\frac{d+(d-2)^2}{d+d^2}$$

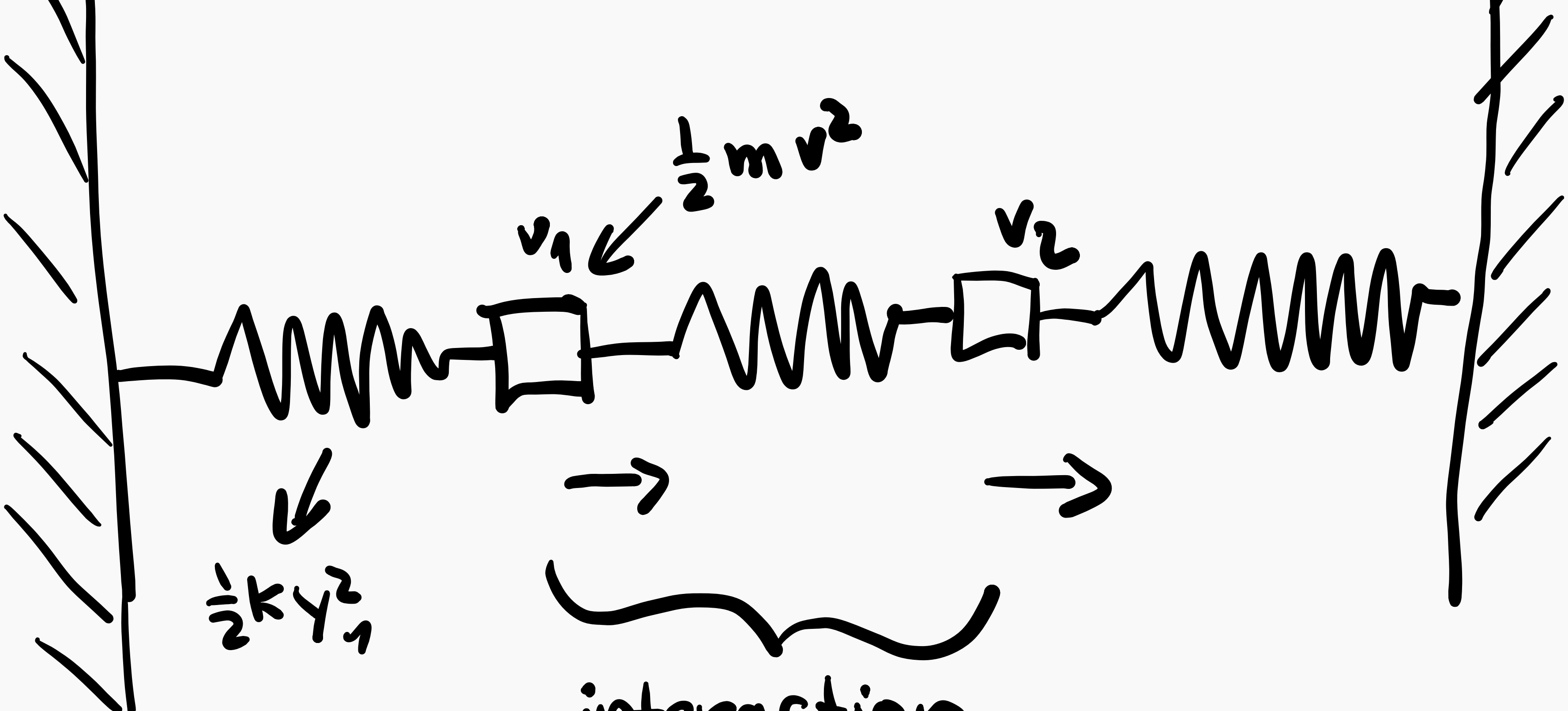


N=Toff



Equations of motion

What will be the system in time t? $F = ma$



interaction between the boxes

Schrödinger equation

linear!

derivative operator

Hamiltonian

how does this quantum system change in time

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H |\psi(t)\rangle \tag{4}$$

vectors

matrix

H can depend on time (but won't in this class).

take $\hbar = 1$

H - describes the energy of a closed system

- hermitian matrix

n qubits

$$i \frac{d}{dt} \begin{pmatrix} \psi_0(t) \\ \psi_1(t) \\ \vdots \\ \psi_d(t) \end{pmatrix} = \begin{pmatrix} H_{00} & H_{01} & & \\ H_{10} & \ddots & & \\ & & \ddots & \\ & & & H_{dd} \end{pmatrix} \begin{pmatrix} \psi_0(t) \\ \vdots \\ \vdots \\ \psi_d(t) \end{pmatrix}$$

2^n rows

time-independent Schrödinger equation

energy = eigenvalue

$$\underbrace{H}_{\text{matrix}}\,|\underbrace{\phi_j(t)}_{\text{eigenstates of } H}\rangle = \overbrace{E_j}^{\text{scalar}}\,|\phi_j(t)\rangle . \tag{5}$$

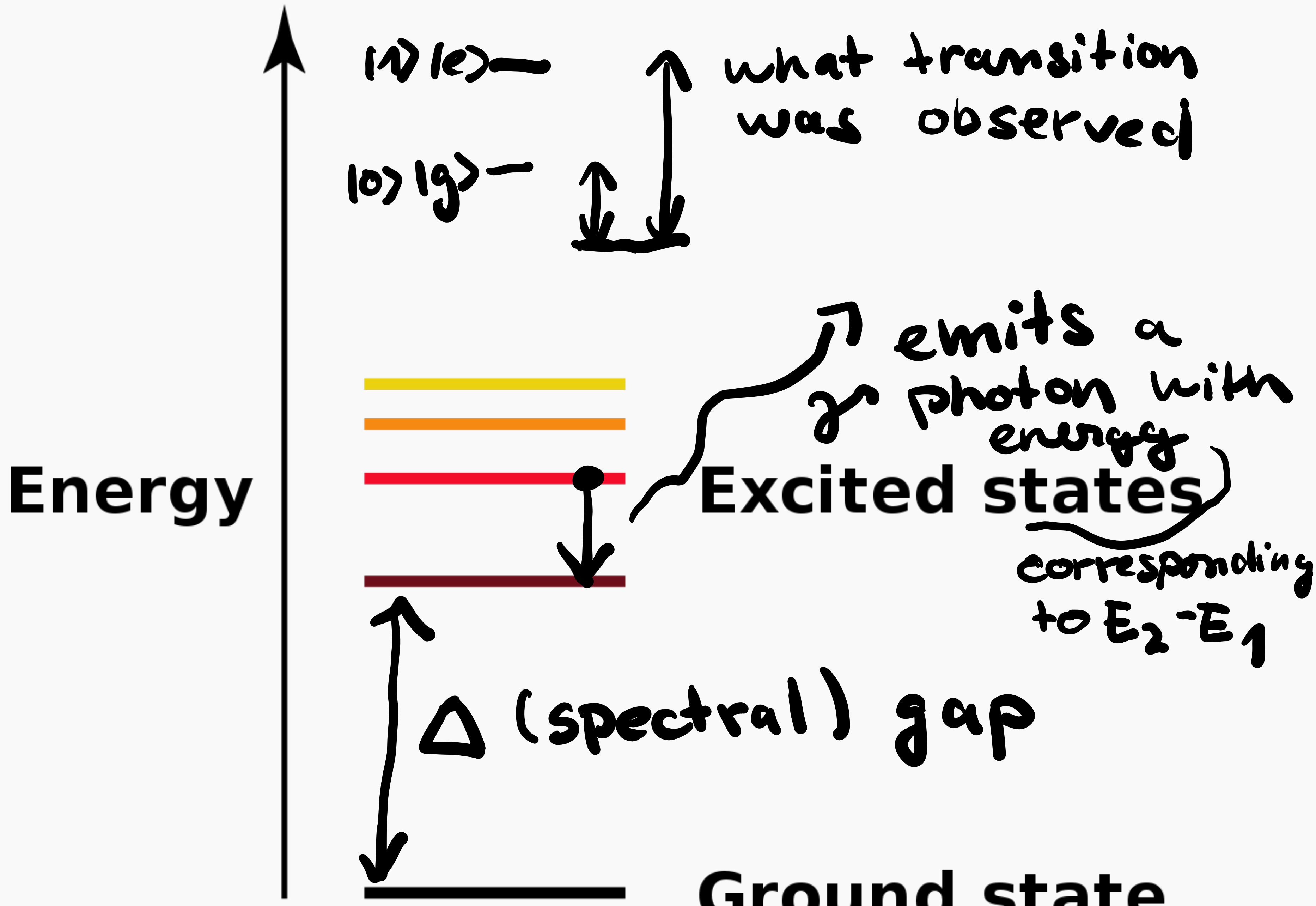
$\{E_j\}$ - spectrum of the Hamiltonian

E_0 - lowest energy, $|\phi_0\rangle$ - ground state
- ground state energy

E_1, E_2 - 1st, 2nd ... excited states

$A\vec{v} = \lambda\vec{v}$

then $A(a\vec{v}) = \lambda a\vec{v}$


Energy
|1⟩ |e⟩
|0⟩ |g⟩
what transition was observed
Excited states
emits a γ photon with energy corresponding to $E_2 - E_1$
Δ (spectral) gap
Ground state

Exercise

Let us take our $H = 2Z$. $\to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ What are the energies are their corresponding eigenstates? Which one is the ground state energy? Verify that for each eigenstate, multiplying it by $e^{-i\alpha}$ for a real α will also yield an eigenstate with the same eigenvalue.

$2\begin{pmatrix} 1 \\ 0 \end{pmatrix} \to$ not an eigenstat (not normalized

$e^{-i\alpha} |v\rangle$

$f(0) = f_0$

number

$$\frac{d}{dt} f(t) = \underbrace{a}_{}\underbrace{f(t)}_{\text{function}} \qquad (6)$$

check if it work

$$\int \frac{df(t)}{f(t)} = \int a\, dt$$

$\ln f = at + c'$

$f(t) = e^{at+c'} = e^{at} \underbrace{e^{c'}}_{c} = Ce^{at}$

$f(0) = f_0 = Ce^0 = C$

$f(t) = f_0 e^{at}$

Formal solution

$$|\psi(t)\rangle = e^{-iHt}\,|\psi(0)\rangle \tag{7}$$

exp. large

$$e^{A} = \sum_{k} \frac{A^{k}}{k!}$$

Simulating Hamiltonian evolution is a hard problem for classical computers (BQP complete)

When $|\psi(0)\rangle = |\phi_j\rangle$ ← eigenstate of H with energy E_j

$$i\frac{d}{dt}|\phi_j(t)\rangle = i\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle = \text{[redacted]}\, E_j|\phi_j\rangle \tag{8}$$

$$\frac{d}{dt}|\phi_j(t)\rangle = -iE_j|\phi_j(t)\rangle$$

$$|\phi_j(t)\rangle = e^{-iE_j}|\phi_j\rangle$$

← eigenstate of H

only picks up this phase

General state

overlaps phases

form a complete basis

Ansatz $|\psi(t)\rangle = \sum_j a_j(t) |\phi_j\rangle$

$\langle \phi_k | \phi_j \rangle = \delta_{jk}$

find the time-dependent coefficients

$$\sum_j \frac{d}{dt}\left(a_j(t) |\phi_j\rangle\right) = -i \sum_j a_j(t) H |\phi_j\rangle =$$

$$= -i \sum_j a_j(t) \overset{E_j}{\cancel{H}} |\phi_j\rangle \quad / \langle \phi_k |$$

$$\frac{d}{dt} a_k(t) = -i\, a_k(t) E_k$$

$$a_k(t) = C_k e^{-itE_k}$$

Initial condition

$$|\Psi(0)\rangle = |\Psi\rangle$$

$$\sum_k c_k \underbrace{e^{-iE_k t}}_{1} |\phi_k\rangle \Big|_{t=0} = |\Psi\rangle$$

$$\sum_k c_k |\phi_k\rangle = |\Psi\rangle \quad / \langle \phi_j |$$

$$c_j = \langle \phi_j | \Psi \rangle$$

Recipe for computing the evolution and problems with it

1. Find the ~~entries~~ energies and eigenstates of the Hamiltonian. Make sure to normalize the eigenstates.
2. Compute the overlaps between the initial state and the eigenstates. We you do this correctly, the overlaps should be normalized. $\sum_k |c_k|^2 = 1$
3. Substitute the energies E_k, eigenstates $|\phi_k\rangle$ and overlaps $\langle\phi_k|\psi(0)\rangle$ to $|\psi(t)\rangle = \sum_k \underbrace{\langle\phi_k|\psi(0)\rangle}_{\text{overlaps}} e^{\overbrace{-iE_k t}^{}}\underbrace{|\phi_k\rangle}_{\text{eigenstates}}$. Sometimes the solution can be further simplified.

phases with energies

Expectation value and energy eigenstates

$$\langle E \rangle = \langle \psi | H | \psi \rangle \,. \tag{9}$$

$$= \mathrm{Tr}(H|\psi\rangle\langle\psi|)$$

$$= \mathrm{Tr}(H\rho)$$

If $|\psi\rangle$ is an energy eigenstate

$\langle E \rangle = \langle \psi | E | \psi \rangle = E$ - eigenvalue

General expectation value and observables

$$\sum_k |\phi_k\rangle\langle\phi_k| = \mathbb{1}$$

not an eigenstate

$$\langle E \rangle = \langle \psi | H | \psi \rangle \,. \tag{10}$$

always reals

$$= \sum_k \langle \psi | H | \phi_k \rangle\langle \phi_k | \psi \rangle$$

E_k

$$= \sum_k E_k \langle \psi | \phi_k \rangle\langle \phi_k | \psi \rangle$$

$$= \sum_k p_k E_k$$

evaluate through sampling

$$p_k = |\langle \psi | \phi_k \rangle|^2$$

POVM $\quad M_k = |\phi_k\rangle\langle\phi_k|$

→ same for any hermitian operator

observables

Exercise

Take a Hamiltonian

$$H = \begin{pmatrix} 3 & 2 \\ 2 & -3 \end{pmatrix} \tag{11}$$

And an initial state

$$|\psi(0)\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{12}$$

Find the evolution of the state $|\psi(t)\rangle$ under H. What will be its expected energy?