Assessment 0 for 41076: Methods in Quantum Computing

This is an optional problem set for students who want to refresh their knowledge of elementary quantum computing and linear algebra.

1. Do the following expressions correspond to normalized quantum states?

   (a) $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}$ ✗

   (b) $\frac{1}{2}|0\rangle + \frac{-1}{2}|1\rangle + \frac{i}{2}|2\rangle + \frac{-i}{2}|3\rangle$ ✓

   (c) $\begin{pmatrix} 0.3 \\ 0.7 \end{pmatrix}$ ✗

a) We cannot add numbers such as $\frac{\sqrt{3}}{2}$ to ket objects like $\frac{1}{2}|0\rangle$. Therefore this is NOT a quantum state. $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$ would correspond to a quantum state because $\left|\frac{1}{2}\right|^2 + \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{1}{4} + \frac{3}{4} = 1$

b) This is a quantum state because $\left|\frac{1}{2}\right|^2 + \left|\frac{-1}{2}\right|^2 + \left|\frac{i}{2}\right|^2 + \left|\frac{-i}{2}\right|^2 = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4}$.
It is a state on two qubits which could be also expressed as
$\frac{1}{2}|00\rangle + \frac{-1}{2}|01\rangle + \frac{i}{2}|10\rangle + \frac{-i}{2}|11\rangle$
in binary notation

c) This vector is not normalized, therefore not a state.
$|0.3|^2 + |0.7|^2 \neq 1$

2. Suppose that we perform a measurement in computational basis on the state $\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. What states can we measure and what would be the probabilities for each measurement?

When we measure in the computational basis, we can detect states $|0\rangle$ and $|1\rangle$. The computational basis is also known as the Z basis and the measurements will correspond to Z's eigenvalues:

$|0\rangle \rightarrow +1$ outcome

$|1\rangle \rightarrow -1$ outcome

The probabilities are $|\frac{1}{2}|^2 = \frac{1}{4}$ for detecting $|0\rangle$ and $|\frac{\sqrt{3}}{2}|^2 = \frac{3}{4}$ for detecting $|1\rangle$.

3. Consider the matrix $X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Which of the following statements are true?

(a) X is an identity. ✗

(b) X is symmetric. ✓

(c) X is diagonal. ✗

(d) X is positive semi-definite. ✗

(e) X is unitary. ✓

(f) X is hermitian. ✓

a) The 2×2 identity matrix is $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, not X.

b) Yes, because $X^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = X$.

c) No, we have non-zero nondiagonal terms.

d) Positive semi-definity matrix has all eigenvalues larger or equal to 0 and real. The eigenvalues of X are 1 and -1. Because of -1, it is not positive semi-definite

e) We can use the fact that a matrix is unitary if and only if its eigenvalues all have magnitude 1 which X has. Or we can notice that $X^{-1} = X^{+} = X$ $(X \cdot X = \mathbb{1})$ therefore it is unitary.

f) Every symmetric matrix is Hermitian. X satisfies $X^{+} = X$.

4. Write the operator $X \otimes I$ as a 4×4 matrix in ~~a~~ the computational basis.

$$X \otimes \mathbb{1} = \left(|0\rangle\langle 1| + |1\rangle\langle 0|\right) \otimes \left(|0\rangle\langle 0| + |1\rangle\langle 1|\right)$$

$$= |00\rangle\langle 10| + |01\rangle\langle 11| + |10\rangle\langle 00| + |11\rangle\langle 01|$$

Each of this 4 terms corresponds to a non-zero matrix element

$$X \otimes \mathbb{1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

We could also examine how the operator transforms basis states:

$$X \otimes \mathbb{1}\,|00\rangle = |10\rangle$$
$$X \otimes \mathbb{1}\,|01\rangle = |11\rangle$$
$$X \otimes \mathbb{1}\,|10\rangle = |00\rangle$$
$$X \otimes \mathbb{1}\,|11\rangle = |01\rangle$$

5. Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

   (a) Compute A^2

   (b) What are A's eigenvalues?

$$A^2 = A \cdot A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

$$A - \lambda \mathbb{1} = \begin{pmatrix} 1-\lambda & 1 \\ 1 & -\lambda \end{pmatrix}$$

$$\det(A - \lambda \mathbb{1}) = 0$$

$$(1-\lambda)(-\lambda) - 1 = 0$$

$$-\lambda + \lambda^2 - 1 = 0$$

$$\lambda = \frac{1 \pm \sqrt{5}}{2}$$

The eigenvalues are $\frac{1 \pm \sqrt{5}}{2}$

The third approach is to write $X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and perform the tensor product.
All approaches lead to the same answer.